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CONCERNING SENSE ON CLOSED CURVES IN NON-METRICAL PLANE ANALYSIS SITUS.[*]

By John Robert Kline.

§ 1. **Introduction.** In a paper recently published in these Annals,[†] I proposed the following non-intuitional definition[‡] for sameness of sense on closed curves in non-metrical plane analysis situs: *The sense* $A_1B_1C_1$[§] *on the closed curve* J_1 *is said to be the same as the sense* $A_2B_2C_2$ *on the closed curve* J_2 *with respect to* E_{12},[||] *if there exists in* E_{12} *a closed curve* J_3 *and three points* A_3, B_3 *and* C_3 *thereon such that* (1) *if it is impossible to join* A_1 *to* A_3, B_1 *to* B_3 *and* C_1 *to* C_3 *by arcs, no two of which have a point in common and which lie except for their endpoints in* E_{13}, *then it is also impossible to join* A_2 *to* A_3, B_2 *to* B_3 *and* C_2 *to* C_3 *by arcs, no two of which have a point in common and which lie except for their endpoints in* E_{23} (2) *if it is possible to join* A_1 *to* A_3, B_1 *to* B_3 *and* C_1 *to* C_3 *as indicated above, then it is also possible to join* A_2 *to* A_3, B_2 *to* B_3 *and* C_2 *to* C_3 *as indicated. Otherwise the sense* $A_1B_1C_1$ *on* J_1 *and the sense* $A_2B_2C_2$ *on* J_2 *are said to be opposite with respect to* E_{12}.

A question might naturally arise whether the sense thus defined coincides with " the ordinary sense." The object of the present paper is: (1) To give a set of three independent postulates or axioms in terms of the undefined symbol "sameness of sense"; (2) to show that, if Σ' is any definition for sameness of sense on plane closed curves, then a necessary and sufficient condition that Σ' should have the properties of Axioms 1 to 3, is that Σ' be equivalent to Σ.

Consider the following axioms for sameness of sense on closed curves in a space satisfying Professor R. L. Moore's system of axioms Σ_3[¶]:

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[*] Presented to the American Mathematical Society, April 26, 1919.

[†] Cf. A Definition of Sense on Closed Curves in non-metrical Plane Analysis Situs, these Annals, 2d series, vol. 19 (1918), pp. 185–200. Hereafter in the present paper, the above paper will be referred to as Definition.

[‡] This definition will be referred to as Definition Σ.

[§] In the present paper, when speaking of the sense BCA on the closed curve J, we shall understand that A, B and C are distinct points of J.

[||] Hereafter in this paper E_i will denote the exterior of the closed curve J_i while I_i denotes its interior. The symbol E_{ik} will denote the common exterior of J_i and J_k.

[¶] Cf. R. L. Moore, "On the foundations of plane analysis situs," Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–64. It should be remembered that Σ_3 is satisfied even by certain spaces which are neither metrical, descriptive or separable.

Axiom 1. *If A and C separate B and D on the closed curve J_1, then the sense ABC on J_1 is not the same as the sense ADC on J_1.*[*]

Axiom 2. *If, on the arc ABC of the closed curve J, E, H and F are points such that, on the arc ABC, the order $AEHFC$ holds while EXF is an arc such that $\underset{\smile}{EXF}$*[†] *is a subset of I, then the sense ABC on J is the same as the sense EHF on $EHFXE$.*

Axiom 3. *If the sense $A_1B_1C_1$ on J_1 is both the same as the sense $A_2B_2C_2$ on J_2 and the same as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3.*

§ 2. **Consequences of Axioms 1–3.** Theorem 1: *If A and B separate C and D on the closed curve J_1 while A_1 and B_1 are points of the arc ACB such that the order AA_1CB_1B holds on ACB, then the sense ACB on J_1 is the same as the sense A_1CB_1 on J_1.*

Proof. Let F and G denote points of the arc ACB such that the order AA_1FCGB_1B holds, while FXG is an arc such that $\underset{\smile}{FXG}$ is a subset of I_1. Then, by Axiom 2, the sense FCG on $FCGXF$ is the same as the sense ACB on J_1 and is also the same as the sense A_1CB_1 on J_1. Hence, by Axiom 3, the sense ACB on J_1 is the same as the sense A_1CB_1 on J_1.

Theorem 2. *If A and C separate B and D on the closed curve J_1, while AXC is an arc such that $\underset{\smile}{AXC}$ is a subset of I_1, then the sense ABC on $ABCXA$ is the same as the sense ABC on J.*

Proof. Let E and H be points of J_1 such that the order $ABCHDEA$ holds. By Theorem 1, the sense EBH on J_1 is the same as the sense ABC on J_1. By Axiom 2, the sense EBH on J_1 is the same as the sense ABC on $ABCXA$. Hence, by Axiom 3, the sense ABC on J_1 is the same as the sense ABC on $ABCXA$.

Theorem 3. *If A and C separate B and D on the closed curve J, then the sense ABC on J is the same as both the sense BCD on J and CDA on J.*

Proof. Let M, O, and N be points of J such that the order $ABMONCDA$ holds while MEN is an arc such that $\underset{\smile}{MEN}$ is a subset of I.

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[*] If A, B and C are distinct points of the closed curve J, we are at liberty to regard the sense ABC with respect to the interior of J or with respect to the exterior of J. However, in the present paper, we shall regard it with respect to the exterior of J although there would be no change in any of our results were we to regard sense always with respect to the interior of J. Let it be further understood in the present paper that the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2" is an abbreviation for the following statement, "The sense $A_1B_1C_1$ on J_1 with respect to E_1, the exterior of J_1, is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_2, the exterior of J_2." It can also be seen that we may conveniently adopt the same convention in Σ where we shall replace the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, the common exterior of J_1 and J_2" by the statement "the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2" interpreted as above.

[†] If ABC is an arc, then the symbol $\underset{\smile}{ABC}$ denotes the point set $ABC - A - B$.

It is clear that on the arc ABC the order $AMONC$ holds while on the arc BCD the order $BMOND$ holds. By Axiom 2 the sense ABC on J is the same as the sense MON on $MONEM$. By Axiom 2, the sense BCD on J is the same as the sense MON on $MONEM$. It follows, by Axiom 3, that the sense ABC on J is the same as the sense BCD on J.

In like manner, the sense BCD on J is the same as the sense CDA on J. Hence by Axiom 3, the sense ABC on J is the same as the sense CDA on J.

THEOREM 4. *If the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12} but is not the same as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ is not the same as the sense $A_3B_3C_3$ on J_3.*

The truth of Theorem 4 is evident at once from Axiom 3 by a "reductio ad absurdum" argument.

THEOREM 5. *If A and C separate D and E on the closed curve J_1, while ABC is an arc such that $\underset{\smile}{ABC}$ is a subset of I_1, then the sense ABC on $ABCDA$ is not the same as the sense ABC on $ABCEA$.*

PROOF. By Theorem 3, the sense ABC on $ABCDA$ and the sense ABC on $ABCEA$ are, respectively, the same as the sense CDA on $ABCDA$ and the sense CEA on $ABCEA$. By Theorem 2, the sense CEA on $ABCEA$ is the same as the sense CEA on J and the sense CDA on $ABCDA$ is the same as the sense CDA on J. It follows, by Axiom 3, that the sense ABC on $ABCDA$ is the same as the sense CDA on J while the sense ABC on $ABCEA$ is the same as the sense CEA on J. Let us suppose Theorem 5 false. Then the sense ABC on $ABCDA$ is the same as the sense ABC on $ABCEA$. It would follow, with the aid of Axiom 3, that the sense CDA on J was the same as the sense CEA on J. But this contradicts Axiom 1. Thus the supposition that our theorem is false, has led to a contradiction.

THEOREM 6. *If A and C separate D and E on the closed curve J while ABC is an arc such that $\underset{\smile}{ABC}$ is a subset of I, then the sense ADC on J is the same as the sense ABC on $ABCEA$.*

PROOF. By Theorem 3, the sense ADC on J is the same as the sense CEA on J. By Theorem 2, the sense CEA on $ABCEA$ is the same as the sense CEA on J. Hence, by Axiom 3, the sense ADC on J is the same as the sense CEA on $ABCEA$. By Theorem 3, the sense ABC on $ABCEA$ is the same as the sense CEA on $ABCEA$. It follows, by Axiom 3, that the sense ADC on J is the same as the sense ABC on $ABCEA$.

THEOREM 7. *Suppose J_1 and J_2 are two simple closed curves such that $J_1 + I_1$ is a subset of E_2. Then a necessary and sufficient condition that* *the sense* $A_1B_1C_1$ *on* J_1 *be not the same as the sense* $A_2B_2C_2$ *on* J_2, *is that it be possible to join* $A_1B_1C_1$ *to* $A_2B_2C_2$ *simply.*[*]

PROOF. (*a*) *The condition is necessary.* There exists at least one pair of arcs A_1XA_2 and C_1ZC_2, which have no point in common and lie except for their endpoints in E_{12}. Suppose the condition were not necessary. Then, for every pair of arcs, A_1XA_2 and C_1ZC_2 described above either (1) I_1 is within and I_2 is without or (2) I_1 is without and I_2 is within $A_1B_1C_1ZC_2B_2A_2XA_1$.[†]

CASE I. Suppose I_1 is within and I_2 without $A_1B_1C_1ZC_2B_2A_2XA_1$. Let D_i ($i = 1, 2$) be a point of the closed curve J_i such that A_i and C_i separate B_i and D_i on J_i. It may easily be proved that the interior of $A_1B_1C_1ZC_2D_2A_2XA_1 = \underparen{A_1D_1C_1} + \underparen{A_2B_2C_2} + I_1 + I_2 +$ the interior of $A_1D_1C_1ZC_2B_2A_2XA_1$. By Theorem 1, the sense $A_1B_1C_1$ on $A_1B_1C_1ZC_2D_2A_2XA_1$ is the same as the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. By Theorem 2, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_1B_1C_1$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$. By Theorem 6, the sense $A_2B_1C_2$ on $A_1B_1C_1ZC_2D_2A_2XA_1$ is the same as the sense $A_2B_2C_2$ on J_2. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2. Thus, in Case I, we are led to a contradiction, if we suppose the condition not necessary.

CASE II. I_2 is within and I_1 without $A_1B_1C_1ZC_2B_2A_2XA_1$. In Case II, we are led to a contradiction if we suppose the condition is not necessary, just as in Case I.

(*b*) *The condition is sufficient.* Suppose $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined. Consider one set of arcs A_1XA_2, B_1YB_2 and C_1ZC_2 no two of which have a point in common and which lie, except for their endpoints entirely in E_{12}. Then I_1 and I_2 are either both within or both without $A_1B_1C_1ZC_2B_2A_2XA_1$.[‡]

CASE I*b*. I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Let D_i [$i = 1, 2$] denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. It may be easily proved that I_1 is within and I_2 is without $A_1D_1C_1ZC_2B_2A_2XA_1$. Hence, by an argument similar in all respects to that given in Case I*a*, it may be proved that the sense $A_1D_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2. By Axiom 1, the sense $A_1D_1C_1$ on J_1

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[*] If A_i, B_i and C_i are distinct points of the closed curve J_i ($i = 1, 2$) where J_1 and J_2 are closed curves such that $J_1 + I_1$ is a subset of E_2, then we shall say that $A_1B_1C_1$ can be *simply joined to* $A_2B_2C_2$, if and only if, there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which lie except for their end points in E_{12}.

[†] Cf. Definition, Theorems *B* and *H*, loc. cit., pp. 187 and 195.

[‡] Cf. Definition, Theorems *B* and *H*, loc. cit., pp. 187–8.

is not the same as the sense $A_2B_2C_2$ on J_2. By Theorem 4, the sense $A_1B_1C_1$ on J_1 is not the same as the sense $A_1B_1C_1$ on J_1.

CASE II*b*. I_1 and I_2 are both within $A_1B_1C_1ZC_2B_2A_2XA_1$. The proof is similar to the proof in Case I*b*.

THEOREM 8. *In order that the sense $A_1B_1C_1$ on the closed curve J_1 be the same as the sense $A_2B_2C_2$ on J_2, it is both necessary and sufficient that there exist in E_{12}, a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_1B_1C_1$ and $A_2B_2C_2$ can both be simply joined to $A_3B_3C_3$.*

PROOF. (*a*) *The condition is necessary.* Let J_3 denote any closed curve such that $J_i + I_i$ $(i = 1, 2)$ is a subset of E_3. On J_3 select four points A_3', B_3', C_3' and D_3' such that A_3' and C_3' separate B_3' and D_3' on J_3. Two cases may arise:

CASE I*a*. $A_1B_1C_1$ and $A_3'B_3'C_3'$ can be simply joined. Then $C_3'B_3'A_3'$ and $A_1B_1C_1$ cannot be simply joined.* Hence, as in the proof of Theorem 7, the sense $C_3'B_3'A_3'$ on J_3 is the same as the sense $A_1B_1C_1$ on J_1. Then it must also be impossible to join $C_3'B_3'A_3'$ and $A_2B_2C_2$ simply. For, if it were possible to join $C_3'B_3'A_3'$ and $A_2B_2C_2$ simply, then by Theorem 7, the sense $A_2B_2C_2$ on J_2 would not be the same as the sense $A_3B_3C_3$ on J_3, which in turn would imply, by Theorem 4, that the sense $A_1B_1C_1$ on J_1 was not the same as the sense $A_2B_2C_2$ on J_2. But this is contrary to hypothesis. Hence $C_3'B_3'A_3'$ and $A_2B_2C_2$ cannot be simply joined. Then $A_2B_2C_2$ and $A_3'B_3'C_3'$ can be simply joined.†

CASE II*a*. $A_1B_1C_1$ and $A_3'B_3'C_3'$ cannot be simply joined. Then it follows as in Case I*a*, that $A_2B_2C_2$ and $A_3'B_3'C_3'$ cannot be simply joined. But then $C_3'B_3'A_3'$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

In Case I(*a*) let $A_3B_3C_3$ denote $A_3'B_3'C_3'$ while in Case II*a* $A_3B_3C_3$ denotes $C_3'B_3'A_3'$.

(*b*) *The condition is sufficient.* Suppose there exists in E_{12} a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$. It follows from the definition of simple joining that $J_i + I_i$ $(i = 1, 2)$ must lie entirely in E_3. As $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$, it follows that $C_3B_3A_3$ can be simply joined to neither $A_1B_1C_1$ nor $A_2B_2C_2$. Then, by Theorem 7, the sense $C_3B_3A_3$ on J_3 is the same as the sense $A_1B_1C_1$ on J_1 and is also the same as the sense $A_2B_2C_2$ on J_2. Hence, by Axiom 3, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2.

THEOREM 9. *Suppose Σ' is any definition of sense in a space satisfying Σ_3. Then a necessary and sufficient condition that Σ' have the properties of Axioms* 1 *to* 3, *is that Σ' be equivalent to Σ.*

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* Cf. Definition, loc. cit., Theorem *K*, pp. 197–8.

† Cf. Definition, Theorem *K*, loc. cit., pp. 197–8.

PROOF. (*a*) *The condition is necessary.* If Axioms 1–3 are satisfied, then Theorems 7 and 8 of the present paper hold and the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 if and only if, there exists in E_{12} a closed curve J_3 and three points A_3, B_3 and C_3 thereon such that $A_3B_3C_3$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

(*b*) *The condition is sufficient.* Suppose Σ'', any definition of sense in Σ_3, is equivalent to Σ. We shall now show that Axioms 1 to 3 are satisfied.

(1) *Axiom 1 is satisfied.* Suppose the points A_1 and C_1 separate B_1 and D_1 on J_1. Choose a simple closed curve J_3 and three points A_3, B_3, and C_3 thereon such that $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined and such that $J_1 + I_1$ is a subset of E_3. Then let A_1ZA_3 and C_1XC_3 be any two arcs having no point in common and lying except for their end-points in E_{13}. Then either I_1 and I_3 are both within or I_1 and I_3 are both without* $A_1B_1C_1XC_3B_3A_3XA_1$. It may easily be proved that either (1) I_1 is without and I_2 within or (2) I_1 is within and I_2 without $A_1D_1C_1X$-$C_3B_3A_3XA_1$. But then $A_3B_3C_3$ and $A_1D_1C_1$ cannot be simply joined. Hence, by Definition, the sense $A_1B_1C_1$ on J_1 is not the same as the sense $A_1D_1C_1$ on J_1.

(2) *Axiom 2 is satisfied.* Suppose (1) that the points A_1 and C_1 separate B_1 and D_1 on the closed curve J_1, (2) M, O and N are points of the arc $A_1B_1C_1$ in the order A_1MONC_1, (3) MQN is an arc such that $\underset{\smile}{MQN}$ is a subset of I_1. Consider the closed curve J_3 and the three points A_3, B_3 and C_3 thereon described in the above proof that Axiom 1 is satisfied. Let $\overline{A}_1\overline{M}$ and $\overline{C}_1\overline{N}$ denote, respectively the arcs of J_1 from A_1 to M and from C_1 to N which fail to contain D_1. Let A_3RM denote the arc $A_1ZA_3 + \overline{A}_1\overline{M}$ while C_3SN denotes the arc $C_1XC_3 + \overline{C}_1\overline{N}$. It follows that either (1) I_3 and the interior of $MONQM$ are both within or (2) the interior of both of these closed curves is without $A_3B_3C_3SNOMRA_3$. Hence $A_3B_3C_3$ and MON can be simply joined. It follows, by Definition, that the sense $A_1B_1C_1$ on J_1 is the same as the sense MON on $MONQM$.

(3) *Axiom 3 is satisfied.* See Theorem 2 of Definition.†

§ 3. **Questions of Independence.** In the following E_i is a definition of sense on closed curves in Σ_3 such that (1) Axiom i is not satisfied, (2) all other axioms of the set 1–3 are satisfied.

E_1. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 for every choice of $A_1B_1C_1$, $A_2B_2C_2$, J_1 and J_2.

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* Cf. Definition, Theorems B and H, loc. cit., pp. 187 and 195.

† Cf. loc. cit., p. 199.

E_2. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 if and only if (1) J_1 and J_2 are identical, (2) A_1, B_1 and C_1 are respectively, identical with A_2, B_2 and C_2.

E_3. The sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2, if and only if (1) the senses are the same according to Σ and (2) J_1 and J_2 have at least one common point.

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